Reply by Authors to D. J. Johns and S. Gopalacharyulu

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ALTHOUGH Bushnell's method[2] for buckling of shells of revolution under non-axisymmetric loads has no basis for comparison with the authors'[1] in the absence of theoretical grounding, the nature of the approximation may be analyzed with a view to rules for users of BOSOR 4. A study is in progress here accordingly, with emphasis on the cases in which the buckling loads according to Ref. 2 deviate considerably lower and higher than their counterparts according to Ref. 1 and the latter method is therefore justified.

The first category is exemplified[3] by an annular plate (inner radius $r = 3.8$, outer radius $R = 41.8$, thickness $t = 0.2$, Poisson's ratio $\nu = 0.3$, and Young's modulus $E = 2\times 10^6$) under a symmetric load, $N_\xi = N_{\xi_0} + N_{\xi_1} * \cos\theta$, at its outer circumference. Boundary conditions on buckling are: inner circumference $U = V = W = M_\xi = 0$; outer circumference $N_\xi = N_{\xi\theta} = W = M_\xi = 0$. Buckling loads λ were examined for different combinations of N_{ξ_1} and N_{ξ_0}. At $N_{\xi_0} = 0$, $N_{\xi_1} = -1$, the result was $\lambda = 13.58$ in Ref. 1 and $\lambda = 9.314$ in Ref. 2, a 31% difference, indicating the exact method.

In the second category, there are numerous cases in which the buckling patterns for axisymmetric and non-axisymmetric loads are completely unrelated, and according to Ref. 2, is quite likely to be higher than its counterpart of Ref. 1. Such a case is the same as the preceding example in which $N_{\xi_0} = -1$ and $N_{\xi_1} = -1$. The result of the eigenvalue λ [$N_{cr} = \lambda(N_{\xi_0} + N_{\xi_1})$] was $\lambda = 7.135$ in Ref. 1 and $\lambda = 14.948$ in Ref. 2.

Another case, for example, is where buckling may occur in axial compression along a meridian, in a long cylinder under a symmetric load (as in Ref. 1), especially if the shell is orthotropic with a large difference between the circumferential and meridional stiffness. (By contrast, under axisymmetric hydrostatic pressure, buckling occurs in tangential compression.)

The buckling load in axisymmetric loading is the same whether determined from the symmetric or the antisymmetric mode; in non-axisymmetric cases with the buckling load differing substantially between the two modes, the solution according to Ref. 2, which is ultimately based on an equivalent axisymmetric prebuckling load, will deviate considerably from the exact one.

For symmetric loads, the buckling mode is either symmetric or antisymmetric, and the buckling load can be determined for each case in order to ascertain which of them yields the lower eigenvalue. This is extremely convenient from the computer time and memory viewpoints, in that instead of considering a single case involving all variables, the symmetric and antisymmetric variables are treated separately in their respective cases. The numerical problems in the authors' paper were worked out accordingly. Concerning the writers' recommendation that only "effective" terms in the Fourier series [Eq. (13) in Ref. 1] be considered, this is unfeasible under the authors' approach (for proof, see Ref. 4).

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Received July 1, 1974; revision received August 8, 1974.
Index categories: Structural Stability Analysis; Structural Static Analysis.

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Comment on "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis"

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THE derivation of the continuum equation for a radial beam rotating normal to the spin axis of its inertially rotating base as presented by Likins, Barbera, and Baddeley,[1] is not entirely satisfying. An alternate slightly generalized development which avoids the effective applied load concept is given below. The notation and coordinate system of Ref. 1 are employed.

As in Ref. 1, consider the radial beam as a solid one in which, under deformation, plane sections remain plane. The displacement at points within the continuum may be represented by

$$u(x, y, x, t) = \xi(y, t) \tag{1}$$

$$v(x, y, z, t) = \zeta(y, t) - x\xi_y(y, t) - z\eta_y(y, t) - \frac{1}{2}\int_0^y \{\xi_y{}^2(y', t) + \eta_y{}^2(y', t)\}\, dy' \tag{2}$$

$$w(x, y, z, t) = \eta(y, t) \tag{3}$$

In Eqs. (1) and (3), displacement is approximated to linear order in $\xi(x, t)$ and $\eta(x, t)$. The axial displacement, $v(x, y, z, t)$, is composed of three components: a) displacement associated with uniform elastic extension of the neutral line [the first term on the right-hand side of Eq. (2)]; b) displacement associated with fiber strain of "plane sections" bending [the second and third terms on the right-hand side of Eq. (2)]; c) displacement associated with the "foreshortening effect" (for derivation see, for example, Appendix B of Ref. 2). It is assumed that the point on the beam's neutral axis at the origin 0 undergoes no elastic displacement. Equation (2) is appropriate to quadratic order in ξ and η. Equations (1–3) are further predicated on the assumption that beam torsion is ignorable; the development may be readily generalized to account for torsion.

Substitution of Eqs. (1–3) into (12–17) of Ref. 1, yields

$$\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0 \tag{4}$$

to a linear order approximation, and

$$\varepsilon_{yy} = (\zeta_y - x\xi_{yy} - z\eta_{yy}) + \tfrac{1}{2}(\zeta_y - x\xi_{yy} - z\eta_{yy})^2 \tag{5}$$

Substitution of Eqs. (4) and (5) into Eq. (11) of Ref. 1 yields, to quadratic order approximation

$$V = (E/2)\int_0^l (A\zeta_y{}^2 + I_z\,\xi_{yy}{}^2 + I_x\,\eta_{yy}{}^2)\, dy \tag{6}$$

Received April 8, 1974.
Index category: Structural Dynamic Analysis.

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where

$$I_z = \iint x^2\,dx\,dz, \quad I_x = \iint z^2\,dx\,dz, \quad A = \iint dx\,dz$$

and

$$\iint x\,dx\,dz = \iint y\,dx\,dz = \iint xy\,dx\,dz = 0$$

Substitution of Eqs. (1–3) into Eq. (6) of Ref. 1 yields

$$T = \frac{1}{2}\int_0^l \{\rho(\dot{\xi}^2+\dot{\zeta}^2+\dot{\eta}^2)+\mu' I_z\dot{\xi}_y{}^2+\mu' I_x\dot{\eta}_y{}^2\}\,dy + (\Omega^2/2)\int_0^l\left[\rho\left\{\xi^2+2y\zeta-y\int_0^y\{(\xi_y{}^2(y',t)+\eta_y{}^2(y',t)\}\,dy'\right\}+\mu' I_z\xi_y{}^2+\mu' I_x\eta_y{}^2\right]dy+\Omega\int_0^l(\rho\xi\dot{\zeta}-\mu' I_z\dot{\zeta}_y)\,dy + \Omega^2\mu'(I_z+l^3/3)/2 \qquad (7)$$

where $\rho = \mu' A$. Equation (7) may be simplified at this point: first, by making the customary assumption that effects of rotary inertia may be ignored (i.e., $I_z = I_x = 0$); and next, by assuming that the neutral line is inextensible which then implies that all terms in ζ and its derivatives may be ignored. Further it may be shown via integration by parts (in a manner similar to that in Ref. 3) that,

$$\int_0^l y\left\{\int_0^y f(y')\,dy'\right\}dy = \frac{1}{2}\int_0^l (l^2-y^2)f(y)\,dy \qquad (8)$$

With Eq. (8) and the above simplifications, Eq. (7) becomes

$$T = \frac{1}{2}\int_0^l \{\rho(\dot{\xi}^2+\dot{\eta}^2)-\tfrac{1}{2}\rho\Omega^2(l^2-y^2)(\xi_y{}^2+\eta_y{}^2)+\rho\Omega^2\xi^2\}\,dy + \Omega^2\mu'\{I_z+l^3/3\}/2 \qquad (9)$$

Equations (6) and (9) may then be combined with Eq. (1) of Ref. 1, to construct motion equations. Routine execution of the variation operation, and integration by parts, leads to

$$EI_z\xi_{yyyy}-\tfrac{1}{2}\rho\Omega^2\{(l^2-y^2)\xi_y\}_y-\rho\Omega^2\xi+\rho\ddot{\xi}=0 \qquad (10)$$

$$EI_x\eta_{yyyy}-\tfrac{1}{2}\rho\Omega^2\{(l^2-y^2)\eta_y\}_y+\rho\ddot{\eta}=0 \qquad (11)$$

Equations (10) and (11) are in accord with Eq. (50) of Ref. 1, and corresponding equations derived by other means in Ref. 4. Further, the concepts underlying Eqs. (1–6) are in accord with established results.[5,6]

In comparing the above derivation with that of Ref. 1, the following points may be noted. The key difference lies in the assumed form for $v(x, y, z, t)$. The required "steady-state solutions about which to linearize oscillatory deformations" of Ref. 1 are avoided herein. The terms in Eqs. (10) and (11) contributing centrifugal stiffening arise quite naturally from the kinetic energy expression. There is no need to resort to the concept of an effective applied load, $P(y)$, or any other artifice. The "cascade of approximations" referred to in Ref. 1 is systematized to a significant extent via a consideration of quadratic-order terms in T and V.

I am not in accord with the author's conclusion that, "except for a small class of very special cases, the continuum mechanics model is barren of useful results," particularly in view of the above discussion. Continuum mechanics techniques similar to those of Ref. 4 or above for handling elastic beams, thin-wall sections, shells, membranes, trusses, etc., are applicable and useful in this class of problems. Further, it is my view that spacecraft will, in the future, be designed with deliberate efforts to ensure structural simplicity in order to maximize reliability, to the point where continuum modeling will be applicable to an appreciable extent.

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Reply by Author to F. R. Vigneron

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WE appreciate Dr. Vigneron's thoughtful comments and constructive criticisms. We agree that he has provided an alternative path to our radial beam result, and that his derivation has the important advantage of providing a more systematic treatment for the beam. We accept his contention that there are other special cases of practical interest, such as simple membranes, plates, and perhaps trusses, for which derivations parallel to his radial beam derivation could be devised.

However, we wish to extend the dialogue with Dr. Vigneron on two specific issues. First, we regret that we gave him the impression that we resort to the concept of an effective applied load as an integral part of our derivation; we left him (and perhaps others) with this impression because in our desire to conform to journal standards of brevity we made a comparison of our results with an established (Meirovitch) textbook treatment in the middle of the textbook derivation, rather than at the true end point established by the equations of motion. The textbook uses the "effective load" concept, and thereby bypasses the use of nonlinear strain-displacement equations (which are used by Dr. Vigneron and in our paper). The essential point of this section of our paper was to illustrate that *without* the distasteful "effective load" concept one must resort to nonlinear strain displacement equations to obtain linear equations of vibration. Once we obtained expressions for kinetic and potential energies without the use of an effective load [see our Eqs. (39) and (45)], we could certainly have solved for the nominal extension v_o and applied the rules of variational calculus to obtain equations of motion from our Eq. (1). This is what we did originally, but in writing the paper we truncated the development by comparing our potential energy in Eq. (45) to that presented in terms of effective load in the Meirovitch text; this made it permissible to record the final equations [Eq. (50)] without further derivation. On this point we hope Dr. Vigneron will accept our rebuttal.

The second issue we must raise will probably remain controversial; this is the role of the "steady-state solutions about which to linearize oscillatory deformations," which is at the heart of our method in the original paper, but which is not employed by Dr. Vigneron. It's important to remember that in our paper we advanced this method as an approach to the most general

Received July 8, 1974.

Index category: Structural Dynamic Analysis.

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